Vertex operator approach for form factors of Belavin's $(\mathbb{Z}/n\mathbb{Z})$-symmetric model

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Dedicated to the memory of my parents

Abstract

Belavin's $(\mathbb{Z}/n\mathbb{Z})$-symmetric model is considered on the basis of bosonization of vertex operators in the $A^{(1)}_{n-1}$ model and vertex-face transformation. Free field representations of nonlocal tail operators are constructed for off diagonal matrix elements with respect to the ground state sectors. As a result, integral formulae for form factors of any local operators in the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model can be obtained, in principle.

1 Introduction

The present paper is a continuation of [1], in which we derived the integral formulae for correlation functions of Belavin's $(\mathbb{Z}/n\mathbb{Z})$-symmetric model [2, 3] on the basis of vertex operator approach [4]. Belavin's $(\mathbb{Z}/n\mathbb{Z})$-symmetric model is an n-state generalization of Baxter's eight-vertex model [5], which has $(\mathbb{Z}/2\mathbb{Z})$-symmetries. As for the eight-vertex model, the integral formulae for correlation functions were derived by Lashkevich and Pugai [6], and those for form factors were derived by Lashkevich [7].

It was found in [6] that the correlation functions of the eight-vertex model can be obtained by using the free field realization of the vertex operators in the eight-vertex SOS model [8], with insertion of the nonlocal opearator Λ, called 'the tail operator'. The most essential part of [6] was the construction of free field representations of Λ's. Furthermore, those of the off-diagoal (with respect to the ground state sector) elements of Λ's were constructed in [7], in order to obtain the form factor formulae of the eight-vertex model.

There are some researches which generalize the study of [6]. The vertex operator approach for higher spin generalization of the eight-vertex model was presented in [9]. For higher rank generalization, the

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integral formulae for correlation functions of Belavin's $(\mathbb{Z}/n\mathbb{Z})$-symmetric model were presented in our previous paper [1]. The expression of the spontaneous polarization of $(\mathbb{Z}/n\mathbb{Z})$-symmetric model [10] was also reproduced in [1], on the basis of vertex operator approach. To the best of our knowledge, there has been no developed research of [7] related to the form factor problem. The aim of the present paper is to give a higher rank generalization of the bosonization scheme in the eight-vertex model.

The present paper is organized as follows. In section 2 we review the basic definitions of $(\mathbb{Z}/n\mathbb{Z})$-symmetric model [2], the corresponding dual face model $A^{(1)}_{n-1}$-model [11], and the vertex-face correspondence. Some detail definitions of the models concerned are listed in Appendix A. In section 3 we introduce the type I and type II vertex operators of both $(\mathbb{Z}/n\mathbb{Z})$-symmetric model and the $A^{(1)}_{n-1}$-model, and also introduce the tail operators. Furthermore, we derive the commutation relations that those operators should satisfy. In order to obtain integral formulae for form factors of $(\mathbb{Z}/n\mathbb{Z})$-symmetric model we construct the free field representations of off-diagonal elements of the tail operators, by using those of the type I [12] and the type II [13, 14] vertex operators in the $A^{(1)}_{n-1}$-model in section 4. Useful operator product expansion (OPE) formulae and commutation relations for basic bosons are given in Appendix B. In section 5 we give some concluding remarks. Among these remarks, a brief proof of the commutation relations of the type I and the type II vertex operators in the $A^{(1)}_{n-1}$-model is given in Appendix C.

2 Basic definitions

The present section aims to formulate the problem, thereby fixing the notation.

2.1 Theta functions

Jacobi theta function with two pseudo-periods 1 and τ ($\mathrm{Im}\,\tau > 0$) are defined as follows:

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (v;\tau) := \sum_{m\in\mathbb{Z}} \exp\left\{\pi\sqrt{-1}(m+a)\ \left[(m+a)\tau + 2(v+b)\right]\right\}, \tag{2.1}$$

for $a, b \in \mathbb{R}$. Let $n \in \mathbb{Z}_{\geqslant 2}$ and $r \in \mathbb{R}$ such that $r > n-1$, and also fix the parameter x such that $0 < x < 1$. We will use the abbreviations,

$$[v] = x^{\frac{v^2}{r}-v}\Theta_{x^{2r}}(x^{2v}), \qquad [v]' = x^{\frac{v^2}{r-1}-v}\Theta_{x^{2r-2}}(x^{2v}), \tag{2.2}$$

where

$$\Theta_q(z) = (z;q)_\infty (qz^{-1};q)_\infty (q;q)_\infty = \sum_{m\in\mathbb{Z}} q^{m(m-1)/2}(-z)^m,$$
$$(z;q_1,\cdots,q_m)_\infty = \prod_{i_1,\cdots,i_m\geqslant 0} (1 - zq_1^{i_1}\cdots q_m^{i_m}).$$

Note that

$$\vartheta \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \left(\frac{v}{r}, \frac{\pi\sqrt{-1}}{\epsilon r}\right) = \sqrt{\frac{\epsilon r}{\pi}} \exp\left(-\frac{\epsilon r}{4}\right)[v],$$

where $x = e^{-\epsilon}$ $(\epsilon > 0)$.

For later conveniences we also introduce the following symbols

$$r_j(v) = z^{\frac{r-1}{r}\frac{n-j}{n}} \frac{g_j(z^{-1})}{g_j(z)}, \quad g_j(z) = \frac{\{x^{2n+2r-j-1}z\}\{x^{j+1}z\}}{\{x^{2n-j+1}z\}\{x^{2r+j-1}z\}}, \tag{2.3}$$

$$r_j^*(v) = z^{\frac{r}{r-1}\frac{n-j}{n}} \frac{g_j^*(z^{-1})}{g_j^*(z)}, \quad g_j^*(z) = \frac{\{x^{2n+2r-j-1}z\}'\{x^{j-1}z\}'}{\{x^{2n-j-1}z\}'\{x^{2r+j-1}z\}'}, \tag{2.4}$$

$$\chi_j(v) = z^{-\frac{j(n-j)}{n}} \frac{\rho_j(z^{-1})}{\rho_j(z)}, \quad \rho_j(z) = \frac{(-x^{2j+1}z; x^2, x^{2n})_\infty(-x^{2n-2j+1}z; x^2, x^{2n})_\infty}{(-xz; x^2, x^{2n})_\infty(-x^{2n+1}z; x^2, x^{2n})_\infty} \tag{2.5}$$

where $z = x^{2v}$, $1 \leqslant j \leqslant n$ and

$$\{z\} = (z; x^{2r}, x^{2n})_\infty, \quad \{z\}' = (z; x^{2r-2}, x^{2n})_\infty. \tag{2.6}$$

In particular we denote $\chi(v) = \chi_1(v)$. These factors will appear in the commutation relations among the type I and type II vertex operators.

The integral kernel for the type I and the type II vertex operators will be given as the products of the following elliptic functions

$$f(v, w) = \frac{[v + \frac{1}{2} - w]}{[v - \frac{1}{2}]}, \quad h(v) = \frac{[v-1]}{[v+1]}, \tag{2.7}$$

$$f^*(v, w) = \frac{[v - \frac{1}{2} + w]'}{[v + \frac{1}{2}]'}, \quad h^*(v) = \frac{[v+1]'}{[v-1]'}. \tag{2.8}$$

In section 4 we use the following identities

$$\sum_{\nu=0}^{n-1} \prod_{\substack{j=0 \\ j\neq\nu}}^{n-1} \frac{f(v_{j+1} - v_j, 1 - p_\nu + p_j)}{[p_\nu - p_j]} = 0, \tag{2.9}$$

and

$$\sum_{\nu=0}^{n-1} \prod_{\substack{j=0 \\ j\neq\nu}}^{n-1} \frac{f^*(v_j - v_{j+1}, 1 - p_j + p_\nu)}{[p_\nu - p_j]'} = 0, \tag{2.10}$$

where $v_n = v + \frac{n}{2}$, and $\sum_{j=0}^{n-1} p_j = 0$. The former one (2.9) was derived in [12] by applying the Liouville's second theorem to the following elliptic function

$$F(w) = \prod_{j=0}^{n-1} \frac{[v_{j+1} - v_j - \frac{1}{2} + w - p_j]}{[v_{j+1} - v_j - \frac{1}{2}][w - p_j]}.$$

The latter one (2.10) can be similarly proved.

2.2 $(\mathbb{Z}/n\mathbb{Z})$-symmetric model and its dual face model

Let $V = \mathbb{C}^n$ and $\{\varepsilon_\mu\}_{0\leqslant\mu\leqslant n-1}$ be the standard orthonormal basis with the inner product $\langle\varepsilon_\mu, \varepsilon_\nu\rangle = \delta_{\mu\nu}$. Belavin's $(\mathbb{Z}/n\mathbb{Z})$-symmetric model [2] is a vertex model on a two-dimensional square lattice $\mathcal{L}$ such that

the state variables take the values of $(\mathbb{Z}/n\mathbb{Z})$-spin. The model is $(\mathbb{Z}/n\mathbb{Z})$-symmetric in a sense that the R-matrix satisfies the following conditions:

$$\begin{aligned} &\text{(i)} \quad R(v)^{ik}_{jl} = 0, \quad \text{unless } i+k = j+l, \ \bmod n, \\ &\text{(ii)} \quad R(v)^{i+pk+p}_{j+pl+p} = R(v)^{ik}_{jl}, \ \forall i,j,k,l,p \in \mathbb{Z}/n\mathbb{Z}. \end{aligned} \tag{2.11}$$

The R-matrix satisfies the Yang-Baxter equation (YBE)

$$R_{12}(v_1 - v_2)R_{13}(v_1 - v_3)R_{23}(v_2 - v_3) = R_{23}(v_2 - v_3)R_{13}(v_1 - v_3)R_{12}(v_1 - v_2), \tag{2.12}$$

where $R_{ij}(v)$ denotes the matrix on $V^{\otimes 3}$, which acts as $R(v)$ on the i-th and j-th components and as identity on the other one. As for the elliptic parametrization of R-matrix, see Appendix A.

The dual face model of $(\mathbb{Z}/n\mathbb{Z})$-symmetric model is called $A^{(1)}_{n-1}$-model. This is a face model on a two-dimensional square lattice $\mathcal{L}^*$, the dual lattice of $\mathcal{L}$, such that the state variables take the values of the dual space of Cartan subalgebra $\mathfrak{h}^*$ of $A^{(1)}_{n-1}$:

$$\mathfrak{h}^* = \bigoplus_{\mu=0}^{n-1} \mathbb{C}\omega_\mu, \tag{2.13}$$

where

$$\omega_\mu := \sum_{\nu=0}^{\mu-1} \bar{\varepsilon}_\nu, \qquad \bar{\varepsilon}_\mu = \varepsilon_\mu - \frac{1}{n}\sum_{\mu=0}^{n-1} \varepsilon_\mu.$$

The weight lattice P and the root lattice Q of $A^{(1)}_{n-1}$ are usually defined. See Appendix A.

An ordered pair $(a,b) \in \mathfrak{h}^{*2}$ is called *admissible* if $b = a + \bar{\varepsilon}_\mu$, for a certain $\mu\,(0 \leqslant \mu \leqslant n-1)$. For $(a,b,c,d) \in \mathfrak{h}^{*4}$ let $W\left[\begin{array}{cc|c} c & d & \\ b & a & v\end{array}\right]$ be the Boltzmann weight of the $A^{(1)}_{n-1}$ model for the state configuration $\left[\begin{array}{cc} c & d \\ b & a \end{array}\right]$ round a face. Here the four states a,b,c and d are ordered clockwise from the SE corner. In this model $W\left[\begin{array}{cc|c} c & d & \\ b & a & v\end{array}\right] = 0$ unless the four pairs $(a,b),(a,d),(b,c)$ and (d,c) are admissible. Non-zero Boltzmann weights are given by (A.6–A.8). See Appendix A.

Among those, the weight (A.7) is different from the corresponding one used in our previous paper [1] by a minus sign. Accordingly, in the present paper we will use different definitions of the intertwining vectors (2.15) and the type I vertex operators (4.5–4.6) from the corresponding objects of [1] by extra factors of the form $(-1)^A$'s. This difference simply results from a gauge transformation.

The Boltzmann weights solve the Yang-Baxter equation for the face model [11]:

$$\begin{aligned} &\sum_g W\left[\begin{array}{cc|c} d & e & \\ c & g & v_1\end{array}\right] W\left[\begin{array}{cc|c} c & g & \\ b & a & v_2\end{array}\right] W\left[\begin{array}{cc|c} e & f & \\ g & a & v_1 - v_2\end{array}\right] \\ = &\sum_g W\left[\begin{array}{cc|c} g & f & \\ b & a & v_1\end{array}\right] W\left[\begin{array}{cc|c} d & e & \\ g & f & v_2\end{array}\right] W\left[\begin{array}{cc|c} d & g & \\ c & b & v_1 - v_2\end{array}\right] \end{aligned} \tag{2.14}$$

2.3 Vertex-face correspondence

Let

$$
\begin{aligned}
t(v)^a_{a-\bar{\varepsilon}_\mu} &= t(v;\epsilon,r)^a_{a-\bar{\varepsilon}_\mu} = \sum_{\nu=0}^{n-1} \varepsilon_\nu t^\nu(v)^a_{a-\bar{\varepsilon}_\mu}, \\
t^\nu(v)^a_{a-\bar{\varepsilon}_\mu} &= \prod_{j=\mu+1}^{n-1} (-1)^{a_{\mu j}} \vartheta \begin{bmatrix} \frac{n}{2} \\ \frac{1}{2}+\frac{\nu}{n} \end{bmatrix} \left(\frac{v}{nr} + \frac{\bar{a}_\mu}{r}; \frac{\pi\sqrt{-1}}{n\epsilon r} \right).
\end{aligned}
\tag{2.15}
$$

be the intertwining vectors. (See Appendix A, concerning the definition of $\bar{a}_\mu$.) Then $t(v)^a_{a-\bar{\varepsilon}_\mu}$'s relate the R-matrix of $(\mathbb{Z}/n\mathbb{Z})$-symmetric model in the principal regime and Boltzmann weights W of $A^{(1)}_{n-1}$-model in the so-called regime III. (cf. figure 1)

$$
R(v_1-v_2)t(v_1)^d_a \otimes t(v_2)^c_d = \sum_b t(v_1)^c_b \otimes t(v_2)^b_a W \left[\begin{array}{cc|c} c & d & \\ & & v_1-v_2 \\ b & a & \end{array} \right]. \tag{2.16}
$$

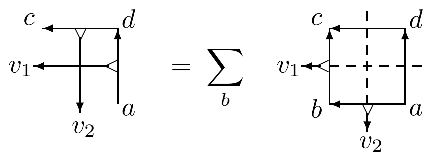


Figure 1. Picture representation of vertex-face correspondence.

Note that the present intertwining vectors are different from the ones used in [11], which relate the R-matrix of $(\mathbb{Z}/n\mathbb{Z})$-symmetric model in the disordered phase and Boltzmann weights W of $A^{(1)}_{n-1}$-model in the regime III.

Let us introduce the dual intertwining vectors (see figure 2) satisfying

$$
\sum_{\mu=0}^{n-1} t^*_\mu(v)^{a'}_a t^\mu(v)^a_{a''} = \delta^{a'}_{a''}, \qquad \sum_{\nu=0}^{n-1} t^\mu(v)^a_{a-\bar{\varepsilon}_\nu} t^*_{\mu'}(v)^{a-\bar{\varepsilon}_\nu}_a = \delta^\mu_{\mu'}. \tag{2.17}
$$

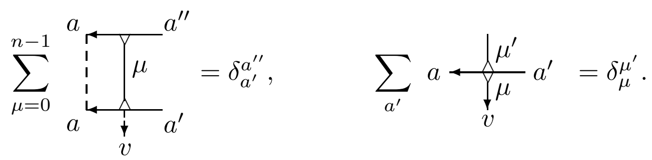


Figure 2. Picture representation of the dual intertwining vectors.

From (2.16) and (2.17), we have (cf. figure 3)

$$
t^*(v_1)^b_c \otimes t^*(v_2)^a_b R(v_1-v_2) = \sum_d W \left[\begin{array}{cc|c} c & d & \\ & & v_1-v_2 \\ b & a & \end{array} \right] t^*(v_1)^a_d \otimes t^*(v_2)^d_c. \tag{2.18}
$$

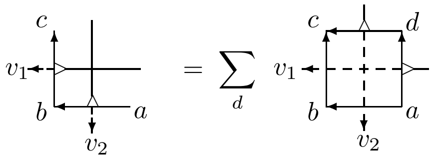


Figure 3. Vertex-face correspondence by dual intertwining vectors.

For fixed $r > n-1$, let

$$S(v) = -R(v)|_{r \mapsto r-1}, \qquad W' \left[\begin{array}{cc|c} c & d & \\ & & v \\ b & a & \end{array} \right] = -W \left[\begin{array}{cc|c} c & d & \\ & & v \\ b & a & \end{array} \right] \Bigg|_{r \mapsto r-1}, \tag{2.19}$$

and

$$t'^*(u)^b_a := t^*(u; \epsilon, r-1)^b_a. \tag{2.20}$$

Then we have

$$t'^*(v_1)^b_c \otimes t'^*(v_2)^a_b S(v_1 - v_2) = \sum_d W' \left[\begin{array}{cc|c} c & d & \\ & & v_1 - v_2 \\ b & a & \end{array} \right] t'^*(v_1)^a_d \otimes t'^*(v_2)^d_c. \tag{2.21}$$

3 Vertex operator algebra

3.1 Vertex operators for $(\mathbb{Z}/n\mathbb{Z})$-symmetric model

Let $\mathcal{H}^{(i)}$ be the $\mathbb{C}$-vector space spanned by the half-infinite pure tensor vectors of the forms:

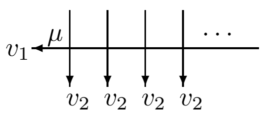
$$\varepsilon_{\mu_1} \otimes \varepsilon_{\mu_2} \otimes \varepsilon_{\mu_3} \otimes \cdots \qquad \text{with } \mu_j \in \mathbb{Z}/n\mathbb{Z},\ \mu_j = i + 1 - j \pmod{n} \text{ for } j \gg 0. \tag{3.1}$$

Let $\mathcal{H}^{*(i)}$ be the dual of $\mathcal{H}^{(i)}$ spanned by the half-infinite pure tensor vectors of the forms

$$\cdots \otimes \varepsilon_{\mu_{-2}} \otimes \varepsilon_{\mu_{-1}} \otimes \varepsilon_{\mu_0} \qquad \text{with } \mu_j \in \mathbb{Z}/n\mathbb{Z},\ \mu_j = i + 1 - j \pmod{n} \text{ for } j \ll 0. \tag{3.2}$$

Introduce the type I vertex operator by the following half-infinite transfer matrix

$$\Phi^\mu(v_1 - v_2) = \tag{3.3}$$



Then the operator (3.3) is an intertwiner from $\mathcal{H}^{(i)}$ to $\mathcal{H}^{(i+1)}$. The type I vertex operators satisfy the following commutation relation:

$$\Phi^\mu(v_1)\Phi^\nu(v_2) = \sum_{\mu',\nu'} R(v_1 - v_2)^{\mu\nu}_{\mu'\nu'} \Phi^{\nu'}(v_2)\Phi^{\mu'}(v_1). \tag{3.4}$$

When we consider an operator related to 'creation-annihilation' process, we need another type of vertex operators, the type II vertex operators that satisfy the following commutation relations:

$$\Psi^*_\nu(v_2)\Psi^*_\mu(v_1) = \sum_{\mu',\nu'} \Psi^*_{\mu'}(v_1)\Psi^*_{\nu'}(v_2) S(v_1 - v_2)^{\mu'\nu'}_{\mu\nu}, \tag{3.5}$$

$$\Phi^{\mu}(v_1)\Psi^*_{\nu}(v_2) = \chi(v_1 - v_2)\Psi^*_{\nu}(v_2)\Phi^{\mu}(v_1). \tag{3.6}$$

Let

$$\rho^{(i)} = x^{2nH_{CTM}} : \mathcal{H}^{(i)} \to \mathcal{H}^{(i)}, \tag{3.7}$$

where H_{CTM} is the CTM Hamiltoian defined in [1]. Then we have the homogeneity relation

$$\Phi^{\mu}(v)\rho^{(i)} = \rho^{(i+1)}\Phi^{\mu}(v-n), \qquad \Psi^*_{\mu}(v)\rho^{(i)} = \rho^{(i+1)}\Psi^*_{\mu}(v-n). \tag{3.8}$$

3.2 Vertex operators for the $A^{(1)}_{n-1}$-model

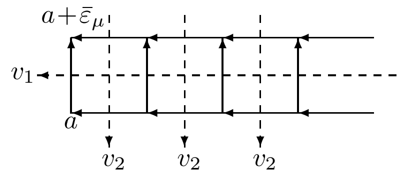
For $k = a+\rho, l = \xi+\rho$ and $0 \leqslant i \leqslant n-1$, let $\mathcal{H}^{(i)}_{l,k}$ be the space of admissible paths $(a_0, a_1, a_2, \cdots)$ such that

$$a_0 = a, \quad a_j - a_{j+1} \in \{\bar{\varepsilon}_0, \bar{\varepsilon}_1, \cdots, \bar{\varepsilon}_{n-1}\}, \text{ for } j = 0,1,2,3,\cdots, \qquad a_j = \xi + \omega_{i+1-j} \text{ for } j \gg 0. \tag{3.9}$$

Also, let $\mathcal{H}^{*(i)}_{l,k}$ be the space of admissible paths $(\cdots, a_{-2}, a_{-1}, a_0)$ such that

$$a_0 = a, \quad a_j - a_{j+1} \in \{\bar{\varepsilon}_0, \bar{\varepsilon}_1, \cdots, \bar{\varepsilon}_{n-1}\}, \text{ for } j = -1,-2,-3,\cdots, \qquad a_j = \xi + \omega_{i+1-j} \text{ for } j \ll 0. \tag{3.10}$$

Introduce the type I vertex operator by the following half-infinite transfer matrix

$\Phi(v_1 - v_2)^{a+\bar{\varepsilon}_\mu}_a =$ 

(3.11)

Then the operator (3.11) is an intertwiner from $\mathcal{H}^{(i)}_{l,k}$ to $\mathcal{H}^{(i+1)}_{l,k+\bar{\varepsilon}_\mu}$. The type I vertex operators satisfy the following commutation relation:

$$\Phi(v_1)^c_b\Phi(v_2)^b_a = \sum_d W\left[\begin{array}{cc|c} c & d & \\ & & v_1 - v_2 \\ b & a & \end{array}\right]\Phi(v_2)^c_d\Phi(v_1)^d_a. \tag{3.12}$$

The free field realization of $\Phi(v_2)^b_a$ was constructed in [12]. See Sec 4.2.

The type II vertex operators should satisfy the following commutation relations:

$$\Psi^*(v_2)^{\xi_c}_{\xi_d}\Psi^*(v_1)^{\xi_d}_{\xi_a} = \sum_{\xi_b}\Psi^*(v_1)^{\xi_c}_{\xi_b}\Psi^*(v_2)^{\xi_b}_{\xi_a}W'\left[\begin{array}{cc|c} \xi_c & \xi_d & \\ & & v_1 - v_2 \\ \xi_b & \xi_a & \end{array}\right], \tag{3.13}$$

$$\Phi(v_1)^{a'}_a\Psi^*(v_2)^{\xi'}_{\xi} = \chi(v_1 - v_2)\Psi^*(v_2)^{\xi'}_{\xi}\Phi(v_1)^{a'}_a. \tag{3.14}$$

Let

$$\rho^{(i)}_{l,k} = G_a x^{2nH^{(i)}_{l,k}}, \tag{3.15}$$

where

$$G_a = \prod_{0\leqslant\mu<\nu\leqslant n-1}[a_{\mu\nu}].$$

Then we have the homogeneity relation

$$\Phi(v)^{a'}_a\frac{\rho^{(i)}_{a+\rho,l}}{G_a} = \frac{\rho^{(i+1)}_{a'+\rho,l}}{G_{a'}}\Phi(v-n)^{a'}_a, \qquad \Psi^*(v)^{\xi'}_{\xi}\rho^{(i)}_{k,\xi+\rho} = \rho^{(i+1)}_{k,\xi'+\rho}\Psi^*(v-n)^{\xi'}_{\xi}. \tag{3.16}$$

The free field realization of $\Psi^*(v)^{\xi'}_{\xi}$ was constructed in [13, 14]. See Sec 4.3.

3.3 Tail operators and commutation relations

In [1] we introduced the intertwining operators between $\mathcal{H}^{(i)}$ and $\mathcal{H}^{(i)}_{l,k}$ ($k = l + \omega_i \pmod{Q}$):

$$
\begin{aligned}
T(u)^{\xi a_0} &= \prod_{j=0}^{\infty} t^{\mu_j}(-u)^{a_j}_{a_{j+1}} : \mathcal{H}^{(i)} \to \mathcal{H}^{(i)}_{l,k}, \\
T(u)_{\xi a_0} &= \prod_{j=0}^{\infty} t^{*}_{\mu_j}(-u)^{a_{j+1}}_{a_j} : \mathcal{H}^{(i)}_{l,k} \to \mathcal{H}^{(i)},
\end{aligned} \tag{3.17}
$$

which satisfy

$$
\rho^{(i)} = \left(\frac{(x^{2r-2}; x^{2r-2})_\infty}{(x^{2r}; x^{2r})_\infty} \right)^{(n-1)(n-2)/2} \frac{1}{G'_\xi} \sum_{\substack{k \equiv l + \omega_i \\ (\mathrm{mod}\ Q)}} T(u)_{a\xi} \rho^{(i)}_{l,k} T(u)^{a\xi}, \tag{3.18}
$$

and the intertwining relations

$$
T(u)^{\xi b} \Phi^{\mu}(v) = \sum_{a} t^{\mu}(v-u)^{b}_{a} \Phi(v)^{b}_{a} T(u)^{\xi a}, \tag{3.19}
$$

$$
T(u)_{\xi b} \Phi(v)^{b}_{a} = \sum_{\mu} t^{*}_{\mu}(v-u)^{a}_{b} \Phi^{\mu}(v) T(u)_{\xi a}. \tag{3.20}
$$

Here, $k = a_0 + \rho$ and $l = \xi + \rho$, and $0 < \Re(u) < \frac{n}{2} + 1$.

In order to obtain the form factors of $(\mathbb{Z}/n\mathbb{Z})$-symmetric model, we need the free field representations of the tail operator which is offdiagonal with respect to the boundary conditions (see figure 4):

$$
\Lambda(u)^{\xi' a'}_{\xi\, a} = T(u)^{\xi' a'} T(u)_{\xi\, a} : \mathcal{H}^{(i)}_{l,k} \to \mathcal{H}^{(i)}_{l' k'}, \tag{3.21}
$$

where $k = a + \rho$, $l = \xi + \rho$, $k' = a' + \rho$, and $l' = \xi' + \rho$. Let

$$
L \left[\begin{array}{cc|c} a'_0 & a'_1 & \\ & & u \\ a_0 & a_1 & \end{array} \right] := \sum_{\mu=0}^{n-1} t^{*}_{\mu}(-u)^{a_1}_{a_0} t^{\mu}(-u)^{a'_0}_{a'_1}. \tag{3.22}
$$

Then we have

$$
\Lambda(u)^{\xi' a'_0}_{\xi\, a_0} = \prod_{j=0}^{\infty} L \left[\begin{array}{cc|c} a'_j & a'_{j+1} & \\ & & u \\ a_j & a_{j+1} & \end{array} \right]. \tag{3.23}
$$

$\Lambda(u)^{\xi' a'_0}_{\xi\, a_0} =$

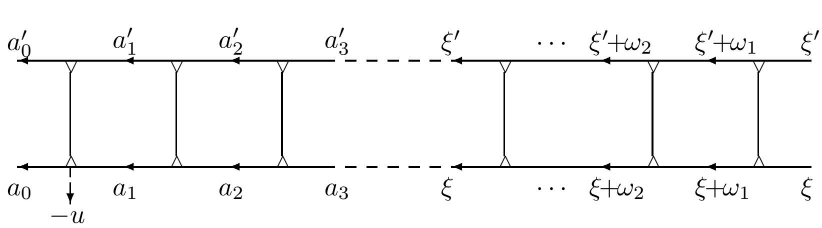


Figure 4. Tail operator $\Lambda(u)^{\xi' a'_0}_{\xi\, a_0}$. The upper (resp. lower) half stands for $T(u)^{\xi' a'_0}$ (resp. $T(u)_{\xi\, a_0}$).

Note that

$$
L \left[\begin{array}{cc|c} a' & a' - \bar{\varepsilon}_\nu & \\ & & u \\ a & a - \bar{\varepsilon}_\mu & \end{array} \right] = \frac{[u + \bar{a}_\mu - \bar{a'}_\nu]}{[u]} \prod_{j \neq \mu} \frac{[\bar{a'}_\nu - \bar{a}_j]}{[a_{\mu j}]}. \tag{3.24}
$$

It is obvious from (2.17), we have

$$L\left[\begin{array}{cc|c} a & a' & \\ a & a'' & u \end{array}\right] = \delta_{a''}^{a'}. \tag{3.25}$$

We therefore have

$$\Lambda(u)_{\xi\, a}^{\xi' a} = \delta_{\xi}^{\xi'}. \tag{3.26}$$

From (3.19), (3.20) and the definition of the tail operator (3.21) we have

$$\Lambda(u)_{\xi\, b}^{\xi' c}\Phi(v)_a^b = \sum_d L\left[\begin{array}{cc|c} c & d & \\ b & a & u-v \end{array}\right]\Phi(v)_d^c\Lambda(u)_{\xi\, a}^{\xi' d}. \tag{3.27}$$

Consider the algebra

$$\Psi^*(v)_{\xi}^{\xi'}T(u)^{\xi a} = \sum_{\mu} T(u)^{\xi' a}\Psi_{\mu}^*(v)t'^{\mu}(v-u-\Delta u)_{\xi}^{\xi'}, \tag{3.28}$$

$$\Psi_{\mu}^*(v)T(u)_{\xi a} = \sum_{\xi'} T(u)_{\xi' a}\Psi^*(v)_{\xi}^{\xi'}t'^*_{\mu}(v-u-\Delta u)_{\xi'}^{\xi}. \tag{3.29}$$

From these, we have

$$\Psi^*(v)_{\xi_d}^{\xi_c}\Lambda(u)_{\xi_a\, a}^{\xi_d\, a'} = \sum_{\xi_b} L'\left[\begin{array}{cc|c} \xi_c & \xi_d & \\ \xi_b & \xi_a & u+\Delta u - v \end{array}\right]\Lambda(u)_{\xi_b\, a}^{\xi_c\, a'}\Psi^*(v)_{\xi_a}^{\xi_b}, \tag{3.30}$$

where

$$L'\left[\begin{array}{cc|c} \xi_c & \xi_d & \\ \xi_b & \xi_a & u \end{array}\right] = L\left[\begin{array}{cc|c} \xi_c & \xi_d & \\ \xi_b & \xi_a & u \end{array}\right]\Bigg|_{r\mapsto r-1}. \tag{3.31}$$

We should find a representation of $\Lambda(u)_{\xi\, a}^{\xi' a'}$ and fix the constant Δu that solves (3.27) and (3.30).

4 Free filed realization

One of the most standard ways to calculate correlation functions and form foctors is the vertex operator approach [4] on the basis of free field representation. The free field representations for the type I vertex operators of the $A_{n-1}^{(1)}$ model were constructed in [12], in terms of oscillators introduced in [15, 16]. Those for the type II vertex operators were constructed in [13, 14], also in terms of oscillators introduced in [15, 16]. It was shown in [17, 18] that the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ provides the Drinfeld realization of the face type elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ tensored by a Heisenberg algebra. Using these representations we derive the free field representation of the tail operator in this section.

4.1 Bosons

Let us consider the bosons B_m^j $(1 \leqslant j \leqslant n-1, m \in \mathbb{Z}\backslash\{0\})$ with the commutation relations

$$[B_m^j, B_{m'}^k] = \begin{cases} m\dfrac{[(n-1)m]_x}{[nm]_x}\dfrac{[(r-1)m]_x}{[rm]_x}\delta_{m+m',0}, & (j=k) \\ -mx^{\mathrm{sgn}(j-k)nm}\dfrac{[m]_x}{[nm]_x}\dfrac{[(r-1)m]_x}{[rm]_x}\delta_{m+m',0}, & (j\neq k), \end{cases} \tag{4.1}$$

where the symbol $[a]_x$ stands for $(x^a - x^{-a})/(x - x^{-1})$. Define B^n_m by

$$\sum_{j=1}^{n} x^{-2jm} B^j_m = 0.$$

Then the commutation relations (4.1) holds for all $1 \leqslant j, k \leqslant n$. These oscillators were introduced in [15, 16].

For $\alpha, \beta \in \mathfrak{h}^*$ let us define the zero mode operators P_α, Q_β with the commutation relations

$$[P_\alpha, \sqrt{-1}Q_\beta] = \langle \alpha, \beta \rangle, \quad [P_\alpha, B^j_m] = [Q_\beta, B^j_m] = 0.$$

We will deal with the bosonic Fock spaces $\mathcal{F}_{l,k}, (l, k \in \mathfrak{h}^*)$ generated by $B^j_{-m} (m > 0)$ over the vacuum vectors $|l, k\rangle$:

$$\mathcal{F}_{l,k} = \mathbb{C}[\{B^j_{-1}, B^j_{-2}, \cdots\}_{1 \leqslant j \leqslant n}]|l, k\rangle,$$

where

$$\begin{aligned}
B^j_m|l, k\rangle &= 0 \ (m > 0), \\
P_\alpha|l, k\rangle &= \langle \alpha, \beta_1 k + \beta_2 l\rangle|l, k\rangle, \\
|l, k\rangle &= \exp\left(\sqrt{-1}(\beta_1 Q_k + \beta_2 Q_l)\right)|0, 0\rangle,
\end{aligned}$$

where β_1 and β_2 are defined by

$$t^2 - \beta_0 t - 1 = (t - \beta_1)(t - \beta_2), \quad \beta_0 = \frac{1}{\sqrt{r(r-1)}}, \quad \beta_1 < \beta_2, \tag{4.2}$$

4.2 Type I vertex operators

Let us define the basic operators for $j = 1, \cdots, n-1$

$$U_{-\alpha_j}(v) = \exp\left(-\beta_1(\sqrt{-1}Q_{\alpha_j} + P_{\alpha_j}\log z)\right) : \exp\left(\sum_{m \neq 0} \frac{1}{m}(B^j_m - B^{j+1}_m)(x^j z)^{-m}\right) :, \tag{4.3}$$

$$U_{\omega_j}(v) = \exp\left(\beta_1(\sqrt{-1}Q_{\omega_j} + P_{\omega_j}\log z)\right) : \exp\left(-\sum_{m \neq 0} \frac{1}{m} \sum_{k=1}^{j} x^{(j-2k+1)m} B^k_m z^{-m}\right) :, \tag{4.4}$$

where $\beta_1 = -\sqrt{\frac{r-1}{r}}$ and $z = x^{2v}$ as usual. For some useful OPE formulae and commutation relations, see Appendix B.

In the sequel we set

$$\pi_\mu = \sqrt{r(r-1)}P_{\bar{\varepsilon}_\mu}, \ \pi_{\mu\nu} = \pi_\mu - \pi_\nu = rL_{\mu\nu} - (r-1)K_{\mu\nu}.$$

The operators $K_{\mu\nu}$, $L_{\mu\nu}$ and $\pi_{\mu\nu}$ act on $\mathcal{F}_{l,k}$ as scalors $\langle \varepsilon_\mu - \varepsilon_\nu, k\rangle$, $\langle \varepsilon_\mu - \varepsilon_\nu, l\rangle$ and $\langle \varepsilon_\mu - \varepsilon_\nu, rl - (r-1)k\rangle$, respectively. In what follows we often use the symbols

$$G_K = \prod_{0 \leqslant \mu < \nu \leqslant n-1} [K_{\mu\nu}], \quad G'_L = \prod_{0 \leqslant \mu < \nu \leqslant n-1} [L_{\mu\nu}]'.$$

For $0 \leqslant \mu \leqslant n-1$ define the type I vertex operator [12] by

$$\phi_\mu(v_0) = \oint_C \prod_{j=1}^{\mu} \frac{dz_j}{2\pi\sqrt{-1}z_j} U_{\omega_1}(v_0)U_{-\alpha_1}(v_1)\cdots U_{-\alpha_\mu}(v_\mu) \prod_{j=0}^{\mu-1} f(v_{j+1}-v_j, K_{j\mu}) \prod_{\substack{j=0\\ j\neq\mu}}^{n-1} [K_{j\mu}]^{-1} \tag{4.5}$$

$$= (-1)^\mu \oint_C \prod_{j=1}^{\mu} \frac{dz_j}{2\pi\sqrt{-1}z_j} U_{-\alpha_\mu}(v_\mu)\cdots U_{-\alpha_1}(v_1)U_{\omega_1}(v_0) \prod_{j=0}^{\mu-1} f(v_j-v_{j+1}, 1-K_{j\mu}) \prod_{\substack{j=0\\ j\neq\mu}}^{n-1} [K_{j\mu}]^{-1}, \tag{4.6}$$

where $z_j = x^{2v_j}$. Considering the factors $f(v_{j+1}-v_j, K_{j\mu})$'s together with the OPE formulae (B.3) and (B.5), the expressions (4.5) has poles at $z_j = x^{\pm(1+2kr)}z_{j-1}\,(k \in \mathbb{Z}_{\geqslant 0})$. The integral contour C for z_j-integration should be chosen such that all integral variables lie in the common convergence domain; i.e., the contour C encircles the poles at $z_j = x^{1+2kr}z_{j-1}\,(k \in \mathbb{Z}_{\geqslant 0})$, but not the poles at $z_j = x^{-1-2kr}z_{j-1}\,(k \in \mathbb{Z}_{\geqslant 0})$.

Note that

$$\phi_\mu(v) : \mathcal{F}_{l,k} \longrightarrow \mathcal{F}_{l,k+\bar{\varepsilon}_\mu}. \tag{4.7}$$

These type I vertex operators satisfy the following commutation relations on $\mathcal{F}_{l,k}$:

$$\phi_{\mu_1}(v_1)\phi_{\mu_2}(v_2) = \sum_{\varepsilon_{\mu_1}+\varepsilon_{\mu_2}=\varepsilon_{\mu'_1}+\varepsilon_{\mu'_2}} W\left[\begin{array}{cc|c} a+\bar{\varepsilon}_{\mu_1}+\bar{\varepsilon}_{\mu_2} & a+\bar{\varepsilon}_{\mu'_1} & \\ & & v_1-v_2 \\ a+\bar{\varepsilon}_{\mu_2} & a & \end{array}\right] \phi_{\mu'_2}(v_2)\phi_{\mu'_1}(v_1). \tag{4.8}$$

We thus denote the operator $\phi_\mu(v)$ by $\Phi(v)_a^{a+\bar{\varepsilon}_\mu}$ on the bosonic Fock space $\mathcal{F}_{l,a+\rho}$.

Dual vertex operators are likewise defined as follows:

$$\begin{aligned} \phi^*_\mu(v) &= (-1)^{n-1-\mu} c_n^{-1} \oint \prod_{j=\mu+1}^{n-1} \frac{dz_j}{2\pi\sqrt{-1}z_j} U_{\omega_{n-1}}\left(v-\frac{n}{2}\right) U_{-\alpha_{n-1}}(v_{n-1})\cdots U_{-\alpha_{\mu+1}}(v_{\mu+1}) \\ &\times \prod_{j=\mu+1}^{n-1} f(v_j-v_{j+1}, K_{\mu j}) \\ &= c_n^{-1} \oint \prod_{j=\mu+1}^{n-1} \frac{dz_j}{2\pi\sqrt{-1}z_j} U_{-\alpha_{\mu+1}}(v_{\mu+1})\cdots U_{-\alpha_{n-1}}(v_{n-1}) U_{\omega_{n-1}}\left(v-\frac{n}{2}\right) \\ &\times \prod_{j=\mu+1}^{n-1} f(v_{j+1}-v_j, 1-K_{\mu j}). \end{aligned} \tag{4.9}$$

Here $v_n = v - \frac{n}{2}$, and

$$c_n = x^{\frac{r-1}{r}\frac{n-1}{2n}} \frac{g_{n-1}(x^n)}{(x^2;x^{2r})^n_\infty (x^{2r};x^{2r})^{2n-3}_\infty},$$

where $g_{n-1}(z)$ is defined by (2.3). The integral contour for z_j-integration encircles the poles at $z_j = x^{1+2kr}z_{j+1}\,(k \in \mathbb{Z}_{\geqslant 0})$, but not the poles at $z_j = x^{-1-2kr}z_{j+1}\,(k \in \mathbb{Z}_{\geqslant 0})$, for $\mu+1 \leqslant j \leqslant n-1$. Note that

$$\phi^*_\mu(v) : \mathcal{F}_{l,k} \longrightarrow \mathcal{F}_{l,k-\bar{\varepsilon}_\mu}. \tag{4.10}$$

The operators $\phi_\mu(v)$ and $\phi^*_\mu(v)$ are dual in the following sense [12]:

$$\sum_{\mu=0}^{n-1} \phi^*_\mu(v)\phi_\mu(v) = 1. \tag{4.11}$$

In [1] we obtained the free field representation of $\Lambda(u)^{\xi\, a'}_{\xi\, a}$ satisfying (3.27) for $\xi'=\xi$:

$$\begin{aligned}\Lambda(u)^{\xi\, a-\bar{\varepsilon}_\mu}_{\xi\, a-\bar{\varepsilon}_\nu} &= G_K \oint \prod_{j=\mu+1}^{\nu} \frac{dz_j}{2\pi\sqrt{-1}z_j} U_{-\alpha_{\mu+1}}(v_{\mu+1})\cdots U_{-\alpha_\nu}(v_\nu) \\ &\times \prod_{j=\mu}^{\nu-1} f(v_{j+1}-v_j, K_{j\nu}) G_K^{-1},\end{aligned} \tag{4.12}$$

where $v_\mu = u$ and $\mu<\nu$.

4.3 Type II vertex operators

Let us define the basic operators for $j=1,\cdots,n-1$

$$V_{-\alpha_j}(v) = \exp\left(-\beta_2(\sqrt{-1}Q_{\alpha_j}+P_{\alpha_j}\log z)\right) : \exp\left(-\sum_{m\neq 0}\frac{1}{m}(A^j_m - A^{j+1}_m)(x^j z)^{-m}\right) :, \tag{4.13}$$

$$V_{\omega_j}(v) = \exp\left(\beta_2(\sqrt{-1}Q_{\omega_j}+P_{\omega_j}\log z\right) : \exp\left(\sum_{m\neq 0}\frac{1}{m}\sum_{k=1}^{j} x^{(j-2k+1)m} A^k_m z^{-m}\right) :, \tag{4.14}$$

where $\beta_2=\sqrt{\frac{r}{r-1}}$ and $z=x^{2v}$, and

$$A^j_m = (-1)^m \frac{[rm]_x}{[(r-1)m]_x} B^j_m. \tag{4.15}$$

For some useful OPE formulae and commutation relations, see Appendix B.

For $0\leqslant \mu \leqslant n-1$ define the type II vertex operator [13, 14][1] by

$$\begin{aligned}\psi^*_\mu(v_0) &= \oint_{C'} \prod_{j=1}^{\mu} \frac{dz_j}{2\pi\sqrt{-1}z_j} V_{\omega_1}(v_0)V_{-\alpha_1}(v_1)\cdots V_{-\alpha_\mu}(v_\mu) \prod_{j=0}^{\mu-1} f^*(v_{j+1}-v_j, L_{j\mu}) &(4.16)\\ &= (-1)^\mu \oint_{C'} \prod_{j=1}^{\mu} \frac{dz_j}{2\pi\sqrt{-1}z_j} V_{-\alpha_\mu}(v_\mu)\cdots V_{-\alpha_1}(v_1)V_{\omega_1}(v_0) \prod_{j=0}^{\mu-1} f^*(v_j-v_{j+1}, 1-L_{j\mu}) &(4.17)\end{aligned}$$

where $z_j = x^{2v_j}$. Considering the factors $f^*(v_{j+1}-v_j, L_{j\mu})$'s together with the OPE formulae (B.9) and (B.11), the expressions (4.16) has poles at $z_j = x^{\pm(-1+2k(r-1))}z_{j-1}$ ($k\in\mathbb{Z}_{\geqslant 0}$). The integral contour C' for z_j-integration should be chosen such that C' encircles the poles at $z_j = x^{-1+2k(r-1)}z_{j-1}$ ($k\in\mathbb{Z}_{\geqslant 0}$), but not the poles at $z_j = x^{1-2k(r-1)}z_{j-1}$ ($k\in\mathbb{Z}_{\geqslant 0}$).

Note that

$$\psi^*_\mu(v) : \mathcal{F}_{l,k} \longrightarrow \mathcal{F}_{l+\bar{\varepsilon}_\mu,k}. \tag{4.18}$$

These type II vertex operators satisfy the following commutation relations on $\mathcal{F}_{l,k}$:

$$\psi^*_{\mu_1}(v_1)\psi^*_{\mu_2}(v_2) = \sum_{\varepsilon_{\mu_1}+\varepsilon_{\mu_2}=\varepsilon_{\mu'_1}+\varepsilon_{\mu'_2}} W'\left[\begin{array}{cc|c} \xi+\bar{\varepsilon}_{\mu_1}+\bar{\varepsilon}_{\mu_2} & \xi+\bar{\varepsilon}_{\mu_2} & \\ \xi+\bar{\varepsilon}_{\mu'_1} & \xi & v_2-v_1\end{array}\right] {\psi^*}_{\mu'_2}(v_2){\psi^*}_{\mu'_1}(v_1). \tag{4.19}$$

[1] Precisely speaking, the integral contour for $\psi^*_\mu(v_0)$ of [13] is different from that of [14]. The contour should be chosen in such a way that all integral variables lie in the convergence domain of the integral formula (4.16). In the present paper we adopt the contour of [14].

We thus denote the operator $\psi^*_\mu(v)$ by $\Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_\mu}$ on the bosonic Fock space $\mathcal{F}_{\xi+\rho,k}$.

Dual vertex operators are likewise defined as follows:

$$\begin{aligned}\psi_\mu(v) &= (-1)^{n-1-\mu} c_n'^{-1} \oint \prod_{j=\mu+1}^{n-1} \frac{dz_j}{2\pi\sqrt{-1}z_j} V_{\omega_{n-1}}\left(v-\frac{n}{2}\right) V_{-\alpha_{n-1}}(v_{n-1})\cdots V_{-\alpha_{\mu+1}}(v_{\mu+1}) \\ &\times \prod_{j=\mu+1}^{n-1} f^*(v_j - v_{j+1}, L_{\mu j}) \prod_{\substack{j=0\\ j\neq\mu}}^{n-1} \frac{[1]'}{[L_{j\mu}]'} \\ &= c_n'^{-1} \oint \prod_{j=\mu+1}^{n-1} \frac{dz_j}{2\pi\sqrt{-1}z_j} V_{-\alpha_{\mu+1}}(v_{\mu+1})\cdots V_{-\alpha_{n-1}}(v_{n-1}) V_{\omega_{n-1}}\left(v-\frac{n}{2}\right) \\ &\times \prod_{j=\mu+1}^{n-1} f^*(v_{j+1}-v_j, 1-L_{\mu j}) \prod_{\substack{j=0\\ j\neq\mu}}^{n-1} \frac{[1]'}{[L_{j\mu}]'}.\end{aligned} \tag{4.20}$$

Here $v_n = v - \frac{n}{2}$, and

$$c_n' = x^{\frac{r(n-1)}{2(r-1)}} \left(\frac{(x^{2r};x^{2r-2})_\infty}{(x^{2r-2};x^{2r-2})_\infty}\right)^n g^*_{n-1}(x^n),$$

where $g^*_{n-1}(z)$ is defined by (2.4). The integral contour for z_j-integration encircles the poles at $z_j = x^{-1+2k(r-1)}z_{j+1}$ ($k \in \mathbb{Z}_{\geqq 0}$), but not the poles at $z_j = x^{1-2k(r-1)}z_{j+1}$ ($k \in \mathbb{Z}_{\geqq 0}$), for $\mu+1 \leqslant j \leqslant n-1$. Note that

$$\psi_\mu(v) : \mathcal{F}_{l,k} \longrightarrow \mathcal{F}_{l-\bar{\varepsilon}_\mu,k}. \tag{4.21}$$

The operators $\psi_\mu(v)$ and $\psi^*_\mu(v)$ are dual in the following sense [14]:

$$\psi_\mu(v)\psi^*_\nu(v') = \delta_{\mu\nu}\frac{1}{1-z'/z} + (\text{regular terms at } v=v'). \tag{4.22}$$

For later convenience, we also introduce another type of basic operators:

$$W_{-\alpha_j}(v) = \exp\left(-\beta_0(\sqrt{-1}Q_{\alpha_j} + P_{\alpha_j}\log(-1)^r z)\right) : \exp\left(-\sum_{m\neq 0}\frac{1}{m}(O^j_m - O^{j+1}_m)(x^j z)^{-m}\right) :, \tag{4.23}$$

where $\beta_0 = \beta_1 + \beta_2 = \dfrac{1}{\sqrt{r(r-1)}}$, $(-1)^r := \exp(\pi\sqrt{-1}r)$ and

$$O^j_m = \frac{[m]_x}{[(r-1)m]_x} B^j_m. \tag{4.24}$$

Concerning useful OPE formulae and commutation relations, see Appendix B.

4.4 Free field realization of tail operators

Consider (3.30) for $(\xi_a, \xi_d, \xi_c) = (\xi, \xi, \xi+\bar{\varepsilon}_{n-1})$, and $(a, a') \to (a+\bar{\varepsilon}_{n-2}, a+\bar{\varepsilon}_{n-1})$:

$$\Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_{n-1}} \Lambda(u)_{\xi\ a+\bar{\varepsilon}_{n-2}}^{\xi\ a+\bar{\varepsilon}_{n-1}} = \sum_{\mu=0}^{n-1} L'\left[\begin{array}{cc|c} \xi+\bar{\varepsilon}_{n-1} & \xi & \\ \xi+\bar{\varepsilon}_\mu & \xi & u+\Delta u - v\end{array}\right] \Lambda(v)_{\xi+\bar{\varepsilon}_\mu\ a+\bar{\varepsilon}_{n-2}}^{\xi+\bar{\varepsilon}_{n-1}\ a+\bar{\varepsilon}_{n-1}} \Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_\mu}. \tag{4.25}$$

This equation can be rewritten as follows:

$$\begin{aligned}&\Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_{n-1}} \Lambda(u)_{\xi\ a+\bar{\varepsilon}_{n-2}}^{\xi\ a+\bar{\varepsilon}_{n-1}} - \Lambda(u)_{\xi+\bar{\varepsilon}_{n-1}\ a+\bar{\varepsilon}_{n-2}}^{\xi+\bar{\varepsilon}_{n-1}\ a+\bar{\varepsilon}_{n-1}} \Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_{n-1}} \\ =\ & \sum_{\mu=0}^{n-2} \frac{[u+\Delta u - v + \xi_{\mu\, n-1}]'}{[u+\Delta u - v]'} \prod_{j\neq\mu} \frac{[\xi_{n-1\, j}+1]'}{[\xi_{\mu j}+1]'} \Lambda(v)_{\xi+\bar{\varepsilon}_\mu\ a+\bar{\varepsilon}_{n-2}}^{\xi+\bar{\varepsilon}_{n-1}\ a+\bar{\varepsilon}_{n-1}} \Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_\mu}.\end{aligned} \tag{4.26}$$

Since the tail operators on the LHS of (4.26) are diagonal components with respect to the ground state sectors, the free field representation (4.12) can be used. Thus, we have

$$\begin{aligned} & \text{LHS of (4.26)} = (-1)^{n-1} G_K \oint_C \frac{dz'}{2\pi\sqrt{-1}z'} \oint_{C'} \prod_{j=1}^{n-1} \frac{dz_j}{2\pi\sqrt{-1}z_j} [V_{-\alpha_{n-1}}(v_{n-1}), U_{-\alpha_{n-1}}(v')] \\ \times \quad & V_{-\alpha_{n-2}}(v_{n-2}) \cdots V_{-\alpha_1}(v_1) V_{\omega_1}(v) f(v'-u, K_{n-2\,n-1}) \prod_{j=0}^{n-2} f^*(v_j - v_{j+1}, 1 - L_{jn-1}) G_K^{-1}, \end{aligned} \tag{4.27}$$

where $z_j = x^{2v_j}$ and $z' = x^{2v'}$. From (B.29) the integral with respect to z_{n-1} of (4.27) can be evaluated by the residues at $z_{n-1} = -x^{\pm 1} z'$. Then the result is

$$\begin{aligned} & \text{LHS}|_{\mathcal{F}_{\xi+\rho, a+\bar{\varepsilon}_{n-2}+\rho}} = \frac{(-1)^{n-1}}{x^{-1}-x} G_K \oint_C \frac{dz'}{2\pi\sqrt{-1}z'} \oint_{C'} \prod_{j=1}^{n-2} \frac{dz_j}{2\pi\sqrt{-1}z_j} \\ \times \quad & \left(F\left(v'+\tfrac{r}{2}\right) W_{-\alpha_{n-1}}\left(v'+\tfrac{r}{2}\right) - F\left(v'-\tfrac{r}{2}\right) W_{-\alpha_{n-1}}\left(v'-\tfrac{r}{2}\right)\right) \\ \times \quad & V_{-\alpha_{n-2}}(v_{n-2}) \cdots V_{-\alpha_1}(v_1) V_{\omega_1}(v) \prod_{j=0}^{n-3} f^*(v_j - v_{j+1}, 1 - \xi_{jn-1}) G_K^{-1} \\ = \quad & \frac{(-1)^n}{x^{-1}-x} G_K \left(\oint_{x^{-r}C} - \oint_{x^r C} \right) \frac{dz'}{2\pi\sqrt{-1}z'} \oint_{C'} \prod_{j=1}^{n-2} \frac{dz_j}{2\pi\sqrt{-1}z_j} F(v') W_{-\alpha_{n-1}}(v') \\ \times \quad & V_{-\alpha_{n-2}}(v_{n-2}) \cdots V_{-\alpha_1}(v_1) V_{\omega_1}(v) \prod_{j=0}^{n-3} f^*(v_j - v_{j+1}, 1 - \xi_{jn-1}) G_K^{-1}, \end{aligned} \tag{4.28}$$

where

$$F(v') = \frac{[v_{n-2} - v' + \frac{r}{2} - \frac{\pi\sqrt{-1}}{2\epsilon} - \xi_{n-2\,n-1}]'}{[v_{n-2} - v' + \frac{r}{2} - \frac{\pi\sqrt{-1}}{2\epsilon}]'} \frac{[v' - u - \frac{r+1}{2} - a_{n-2\,n-1}]}{[v' - u - \frac{r+1}{2}]}. \tag{4.29}$$

The integral with respect to z' of (4.28) can be evaluated by the residues at $z' = -x^r z_{n-2}$ and $z' = x^{-r+1+2u}$. The former residue vanishes because of (B.40)[2]. Thus we have

$$\begin{aligned} (4.28) \quad = \quad & (-1)^n \oint_{C'} \prod_{j=1}^{n-2} \frac{dz_j}{2\pi\sqrt{-1}z_j} W_{-\alpha_{n-1}}\left(u - \tfrac{r-1}{2}\right) V_{-\alpha_{n-2}}(v_{n-2}) \cdots V_{-\alpha_1}(v_1) V_{\omega_1}(v) \\ \times \quad & \prod_{j=0}^{n-2} f^*(v_j - v_{j+1}, 1 - \xi_{j\,n-1}) \frac{[a_{n-2\,n-1}]}{(x^{-1}-x)(x^{2r};x^{2r})_\infty^3} \frac{G_{a+\bar{\varepsilon}_{n-1}}}{G_{a+\bar{\varepsilon}_{n-2}}}. \end{aligned} \tag{4.30}$$

On (4.30), we should read as $v_{n-1} = u + \frac{\pi\sqrt{-1}}{2\epsilon}$. Equating (4.30) and the RHS of (4.26) and using the identity (2.10), we find the free filed representation of the tail operator

$$\begin{aligned} & \Lambda(v)^{\xi+\bar{\varepsilon}_{n-1}\, a+\bar{\varepsilon}_{n-1}}_{\xi+\bar{\varepsilon}_\mu\, a+\bar{\varepsilon}_{n-2}} = \frac{(-1)^{n-\mu}[a_{n-2\,n-1}]}{(x^{-1}-x)(x^{2r};x^{2r})_\infty^3} \frac{[\xi_{\mu\,n-1} - 1]'}{[1]'} G_K {G'_L}^{-1} \\ \times \quad & \oint_{C'} \prod_{j=\mu+1}^{n-2} \frac{dz_j}{2\pi\sqrt{-1}z_j} W_{-\alpha_{n-1}}\left(u - \tfrac{r-1}{2}\right) V_{-\alpha_{n-2}}(v_{n-2}) \cdots V_{-\alpha_{\mu+1}}(v_{\mu+1}) \\ \times \quad & \prod_{j=\mu+1}^{n-2} f^*(v_j - v_{j+1}, L_{\mu j}) G_K^{-1} G'_L, \end{aligned} \tag{4.31}$$

for $0 \leqslant \mu \leqslant n-2$ with $\Delta u = -\frac{n-1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}$ and $v_{n-1} = u + \frac{\pi\sqrt{-1}}{2\epsilon}$.

[2] When $n = 2$ we use (B.41).

Let us return to eq. (3.30) with $\Delta u = -\frac{n-1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}$. By taking an appropriate linear combination of (3.30), we have the following relation:

$$\sum_{\mu=0}^{n-1} A_\mu \Psi^*(v)_{\xi'-\bar{\varepsilon}_\mu}^{\xi'} \Lambda(u)_{\xi\ a}^{\xi'-\bar{\varepsilon}_\mu\ a'} = B\Lambda(u)_{\xi+\bar{\varepsilon}_0\ a}^{\xi'\ a'} \Psi^*(v)_{\xi}^{\xi+\bar{\varepsilon}_0}. \tag{4.32}$$

Here, the coefficients are

$$\begin{aligned} A_\mu &= \prod_{\substack{j=0\\ j\neq\mu}}^{n-1} \frac{1}{[\xi'_{\mu j}]'} \frac{[u-v-\frac{n-1}{2}+\frac{\pi\sqrt{-1}}{2\epsilon}+\bar{\xi}'_\mu-\bar{\xi}_0+\frac{1}{n}]'}{[u-v-\frac{n-1}{2}+\frac{\pi\sqrt{-1}}{2\epsilon}+\bar{\xi}'_0-\bar{\xi}_0+\frac{1}{n}]'} \frac{[\bar{\xi}'_0-\bar{\xi}_0+\frac{1}{n}]'}{[\bar{\xi}'_\mu-\bar{\xi}_0+\frac{1}{n}]'}, \\ B &= \frac{[u-v-\frac{n-3}{2}+\frac{\pi\sqrt{-1}}{2\epsilon}]'}{[u-v-\frac{n-3}{2}+\frac{\pi\sqrt{-1}}{2\epsilon}+\bar{\xi}'_0-\bar{\xi}_0+\frac{1}{n}]'} \prod_{j=1}^{n-1} \frac{[\xi'_{j0}]'}{[\bar{\xi}'_j-\bar{\xi}_0+\frac{1}{n}]'[\xi_{0j}+1]'}. \end{aligned}$$

Consider the product

$$\begin{aligned} & V_{\omega_1}(v)V_{-\alpha_1}(v_1)\cdots V_{-\alpha_{n-2}}(v_{n-2})W_{-\alpha_{n-2}}\left(u-\tfrac{r-1}{2}\right) \\ =\ & :V_{\omega_1}(v)V_{-\alpha_1}(v_1)\cdots V_{-\alpha_{n-2}}(v_{n-2})W_{-\alpha_{n-2}}\left(u-\tfrac{r-1}{2}\right): \\ \times\ & \prod_{j=1}^{n-2} z_{j-1}^{-\frac{r}{r-1}} \frac{(x^{2r-1}\frac{z_j}{z_{j-1}};x^{2r-2})_\infty}{(x^{-1}\frac{z_j}{z_{j-1}};x^{2r-2})_\infty} \cdot z_{n-2}^{-\frac{1}{r-1}} \frac{(-x\frac{x^{2u}}{z_{n-2}};x^{2r-2})_\infty}{(-x^{-1}\frac{x^{2u}}{z_{n-2}};x^{2r-2})_\infty}. \end{aligned} \tag{4.33}$$

The convergence domain of (4.33) is that $x^{-1}|z_j| < |z_{j-1}|$ $(1 \leqslant j \leqslant n-2)$ and $|-x^{2u-1}| < |z_{n-2}|$. Thus, each term of the LHS of (4.32) has a pole at $z = -x^{1-n}x^{2u}$ $(v = u - \frac{n-1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon})$ because pinching occurs at the pole. On the other hand, the RHS of (4.32) does not have such a pole. Hence the singularities at $v = u - \frac{n-1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}$ on the RHS of (4.32) cancel each other:

$$\sum_{\mu=0}^{n-1}\prod_{\substack{j=0\\ j\neq\mu}}^{n-1} \frac{1}{[\xi'_{\mu j}]'} \Psi^*\left(u-\tfrac{n-1}{2}+\tfrac{\pi\sqrt{-1}}{2\epsilon}\right)_{\xi'-\bar{\varepsilon}_\mu}^{\xi'} \Lambda(u)_{\xi\ a}^{\xi'-\bar{\varepsilon}_\mu\ a'} = O(1). \tag{4.34}$$

From (4.34) and (2.10) we find the representation

$$\Lambda(u)_{\xi\ a}^{\xi'-\bar{\varepsilon}_\mu\ a'} = \oint_{C'} \prod_{j=\mu+1}^{n-1} f^*(v_j-v_{j+1},L_{j\mu}) \frac{dz_j}{2\pi\sqrt{-1}z_j} V_{-\alpha_{\mu+1}}(v_{\mu+1})\cdots V_{-\alpha_{n-1}}(v_{n-1}) \cdot \Lambda(u)_{\xi\ a}^{\xi'-\bar{\varepsilon}_{n-1}\ a'}, \tag{4.35}$$

where $v_n = u + \frac{1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}$.

In a similar way to derive (4.32) from (3.30), we can derive the following relation from (3.27):

$$\begin{aligned} & \sum_{\mu=0}^{n-1} \Lambda(u)_{\xi\ a+\bar{\varepsilon}_\mu}^{\xi'\ a'} \Phi(v)_a^{a+\bar{\varepsilon}_\mu} \prod_{\substack{j=0\\ j\neq\mu}}^{n-1} \frac{[a_{\mu j}+1]}{[a_{\mu j}]} [u-v+\bar{a}'_\nu-\bar{a}_\mu+\tfrac{1}{n}] \prod_{\substack{j=0\\ j\neq\nu}}^{n-1} [\bar{a}'_j-\bar{a}_\mu+\tfrac{1}{n}] \\ =\ & [u-v+1]\prod_{\substack{j=0\\ j\neq\nu}}^{n-1} [a'_{\nu j}]\Phi(v)_{a'-\bar{\varepsilon}_\nu}^{a'} \Lambda(u)_{\xi\ a}^{\xi'\ a'-\bar{\varepsilon}_\nu}. \end{aligned} \tag{4.36}$$

Let $v = u+1$ and take the sum over $0 \leqslant \nu \leqslant n-1$. Then we have

$$\sum_{\mu=0}^{n-1} A_\mu(a,a')\Lambda(u)_{\xi\ a+\bar{\varepsilon}_\mu}^{\xi'\ a'} \Phi(u+1)_a^{a+\bar{\varepsilon}_\mu} \prod_{\substack{j=0\\ j\neq\mu}}^{n-1} \frac{[(a+\bar{\varepsilon}_\mu)_{\mu j}]}{[a_{\mu j}]} = 0, \tag{4.37}$$

where

$$A_\mu(a,a')=\sum_{\nu=0}^{n-1}\prod_{j=0}^{n-1}[(a'-\bar{\varepsilon}_\nu)_j-\bar{a}_\mu].$$

From (4.37) and (2.9), we obtain the expression

$$\begin{aligned}\Lambda(u)_{\xi\ a+\bar{\varepsilon}_\mu}^{\xi'\ a'} &= \Lambda(u)_{\xi\ a+\bar{\varepsilon}_{n-1}}^{\xi'\ a'}(-1)^{n-1-\mu}G_K\oint_C\prod_{j=\mu+1}^{n-1}\frac{dz_j}{2\pi\sqrt{-1}z_j}U_{-\alpha_{n-1}}(v_{n-1})\cdots U_{-\alpha_{\mu+1}}(v_{\mu+1})\\ &\times\prod_{j=\mu+1}^{n-1}f(v_j-v_{j+1},K_{\mu\,j})G_K^{-1}\frac{A_{n-1}(a,a')}{A_\mu(a,a')},\end{aligned}\tag{4.38}$$

where $v_n=u-\frac{n-2}{2}$.

Combining eqs. (4.31, 4.35, 4.38), we can construct a free field representations of any $\Lambda(u)_{\xi\ a}^{\xi'\ a'}$, in principle.

4.5 Form factors

Form factors of $(\mathbb{Z}/n\mathbb{Z})$-symmetric model are defined as matrix elements of some local operators. Consider the local operator

$$\mathcal{O}=E^{(1)}_{\mu_1\mu'_1}\cdots E^{(N)}_{\mu_N\mu'_N},\tag{4.39}$$

where $E^{(j)}_{\mu_j\mu'_j}$ is the matrix unit on the j-th site. The free field representation of $\mathcal{O}$ is given by

$$\hat{\mathcal{O}}=\Phi^*_{\mu_1}(u_1)\cdots\Phi^*_{\mu_N}(u_N)\Phi^{\mu'_N}(u_N)\cdots\Phi^{\mu'_1}(u_1).\tag{4.40}$$

The corresponding form factors with m ‘charged’ particles are given by

$$F_m^{(i)}(\mathcal{O};v_1,\cdots,v_m)_{\nu_1\cdots\nu_m}=\frac{1}{\chi^{(i)}}\mathrm{Tr}_{\mathcal{H}^{(i)}}\left(\Psi^*_{\nu_1}(v_1)\cdots\Psi^*_{\nu_m}(v_m)\hat{\mathcal{O}}\rho^{(i)}\right),\tag{4.41}$$

where

$$\chi^{(i)}=\mathrm{Tr}_{\mathcal{H}^{(i)}}\,\rho^{(i)}=\frac{(x^{2n};x^{2n})_\infty}{(x^2;x^2)_\infty}.\tag{4.42}$$

and $m\equiv 0$ (mod n). Note that the local operator (4.39) commute with the type II vertex operators because of (4.40) and (3.6).

By using (3.18), (3.28) and (3.19), we can rewrite (4.41) as follows:

$$\begin{aligned}&F_m^{(i)}(\mathcal{O};v_1,\cdots,v_m)_{\nu_1\cdots\nu_m}\\ =\ &\frac{1}{\chi^{(i)}}\sum_{\xi_1,\cdots,\xi_m}t'^*_{\nu_1}\left(v_1-u+\tfrac{n-1}{2}-\tfrac{\pi\sqrt{-1}}{2\epsilon}\right)_{\xi}^{\xi_1}\cdots t'^*_{\nu_m}\left(v_m-u+\tfrac{n-1}{2}-\tfrac{\pi\sqrt{-1}}{2\epsilon}\right)_{\xi_{m-1}}^{\xi_m}\\ \times\ &\sum_{\substack{k\equiv l+\omega_i\\ (\mathrm{mod}\ Q)}}\sum_{\substack{a_1\cdots a_N\\ a'_1\cdots a'_N}}t^*_{\mu_1}(u_1-u)^a_{a_1}\cdots t^*_{\mu_N}(u_N-u)^{a_{N-1}}_{a_N}t^{\mu'_N}(u_N-u)^{a_N}_{a'_N}\cdots t^{\mu'_1}(u_1-u)^{a'_2}_{a'_1}\\ \times\ &\mathrm{Tr}_{\mathcal{H}^{(i)}_{l,k}}\left(\Psi^*(v_1)^{\xi}_{\xi_1}\cdots\Psi^*(v_m)^{\xi_{m-1}}_{\xi_m}\Phi^*(u_1)^a_{a_1}\cdots\Phi^*(u_N)^{a_{N-1}}_{a_N}\Phi(u_N)^{a_N}_{a'_N}\cdots\Phi(u_1)^{a'_2}_{a'_1}\Lambda(u)^{\xi_m\,a'_1}_{\xi\ a}\frac{\rho^{(i)}_{l,k}}{b_l}\right),\end{aligned}\tag{4.43}$$

where $k=a+\rho$, $l=\xi+\rho$, and

$$b_l=\left(\frac{(x^{2r};x^{2r})_\infty}{(x^{2r-2};x^{2r-2})_\infty}\right)^{(n-1)(n-2)/2}G'_\xi.\tag{4.44}$$

Free filed representations of the tail operators Λ's have been constructed in the present paper, besides all other operators Φ's, Φ^*'s and Ψ^*'s on (4.43) were given in [12, 14, 1]. Integral formulae can be therefore obtained for form factors of $(\mathbb{Z}/n\mathbb{Z})$-symmetric model, in principle.

5 Concluding remarks

In this paper we present vertex operator approach for form factors of $(\mathbb{Z}/n\mathbb{Z})$-symmetric model. For that purpose we constructed the free field representations of the tail operators $\Lambda_{\xi\,a}^{\xi'a'}$, the nonlocal operators which relate the physical quantities of $(\mathbb{Z}/n\mathbb{Z})$-symmetric model and $A_{n-1}^{(1)}$-model. As a result, we can obtain the integral formulae for form factors of $(\mathbb{Z}/n\mathbb{Z})$-symmetric model, in principle.

Our approach is based on some assumptions. We assumed that the vertex operator algebra (3.18–3.20) and (3.28–3.29) correctly describes the intertwining relation between $(\mathbb{Z}/n\mathbb{Z})$-symmetric model and $A_{n-1}^{(1)}$-model. We also assumed that the free field representations (4.31, 4.35, 4.38) provide relevant representations of the vertex operator algebra. As a consistency check of our bosonization scheme, it is thus important to derive closed expressions for form factors of some simple local operators by performing the integrals on (4.43). We wish to address the problem in a separate paper.

Before ending the present paper, we should add one thing. In order to find the free field representations of the tail operators (4.31), we used the correct commutation relation (B.29). In our previous paper [14] we proved (3.14) by using the commutativity of $U_{-\alpha_j}(v)$ and $V_{-\alpha_j}(v')$, instead of (B.29). In Appendix C we thus prove (3.14) on the basis of (B.29).

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A Appendix A Definitions of the models concerned

A.1 Belavin's vertex model

In the original papers [2, 3], the R-matrix in the disordered phase is given. For the present purpose, we need the following R-matrix:

$$\begin{aligned} R(v) &= \frac{[1]}{[1-v]}r_1(v)\overline{R}(v), \\ \overline{R}(v)^{ik}_{jl} &= \frac{h(v)\vartheta\left[\begin{array}{c}\frac{1}{2}\\ \frac{1}{2}+\frac{k-i}{n}\end{array}\right]\left(\frac{1-v}{nr};\frac{\pi\sqrt{-1}}{n\epsilon r}\right)}{\vartheta\left[\begin{array}{c}\frac{1}{2}\\ \frac{1}{2}+\frac{j-k}{n}\end{array}\right]\left(\frac{v}{nr};\frac{\pi\sqrt{-1}}{n\epsilon r}\right)\vartheta\left[\begin{array}{c}\frac{1}{2}\\ \frac{1}{2}+\frac{j-i}{n}\end{array}\right]\left(\frac{1}{nr};\frac{\pi\sqrt{-1}}{n\epsilon r}\right)}\delta^{i+k}_{j+l\ (\mathrm{mod}\, n)}, \end{aligned} \tag{A.1}$$

where $r_1(v)$ is defined by (2.3), and

$$h(v)=\prod_{j=0}^{n-1}\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\frac{j}{n}\end{bmatrix}\left(\frac{v}{nr};\frac{\pi\sqrt{-1}}{n\epsilon r}\right)\Big/\prod_{j=1}^{n-1}\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\frac{j}{n}\end{bmatrix}\left(0;\frac{\pi\sqrt{-1}}{n\epsilon r}\right).$$

We assume that the parameters v, ϵ and r lie in the so-called principal regime:

$$\epsilon>0,\quad r>n-1,\quad 0<v<1. \tag{A.2}$$

Note that the weights (A.1) reproduce those of the eight-vertex model in the principal regime when $n=2$ [5].

A.2 The weight lattice and the root lattice of $A_{n-1}^{(1)}$

Let $V=\mathbb{C}^n$ and $\{\varepsilon_\mu\}_{0\leqslant\mu\leqslant n-1}$ be the standard orthonormal basis as before. The weight lattice of $A_{n-1}^{(1)}$ is defined as follows:

$$P=\bigoplus_{\mu=0}^{n-1}\mathbb{Z}\bar{\varepsilon}_\mu, \tag{A.3}$$

where

$$\bar{\varepsilon}_\mu=\varepsilon_\mu-\varepsilon,\qquad \varepsilon=\frac{1}{n}\sum_{\mu=0}^{n-1}\varepsilon_\mu.$$

We denote the fundamental weights by ω_μ $(1\leqslant\mu\leqslant n-1)$

$$\omega_\mu=\sum_{\nu=0}^{\mu-1}\bar{\varepsilon}_\nu,$$

and also denote the simple roots by α_μ $(1\leqslant\mu\leqslant n-1)$

$$\alpha_\mu=\varepsilon_{\mu-1}-\varepsilon_\mu=\bar{\varepsilon}_{\mu-1}-\bar{\varepsilon}_\mu.$$

The root lattice of $A_{n-1}^{(1)}$ is defined as follows:

$$Q=\bigoplus_{\mu=1}^{n-1}\mathbb{Z}\alpha_\mu, \tag{A.4}$$

For $a\in P$ we set

$$a_{\mu\nu}=\bar{a}_\mu-\bar{a}_\nu,\qquad \bar{a}_\mu=\langle a+\rho,\varepsilon_\mu\rangle=\langle a+\rho,\bar{\varepsilon}_\mu\rangle,\qquad \rho=\sum_{\mu=1}^{n-1}\omega_\mu. \tag{A.5}$$

In this paper we admit not only the case $a\in P$ but also the case $a\in\mathfrak{h}^*:=\mathbb{C}\omega_0\oplus\mathbb{C}\omega_1\oplus\cdots\oplus\mathbb{C}\omega_{n-1}$. For $r>n-1$, let $\sum_{\mu=0}^{n-1}k^\mu=r$, where $a+\rho=\sum_{\mu=0}^{n-1}k^\mu\omega_\mu$, then we denote $a\in\mathfrak{h}^*_{r-n}$.

A.3 The $A_{n-1}^{(1)}$ face model

An ordered pair $(a,b) \in {\mathfrak{h}^*}^2_{r-n}$ is called *admissible* if $b = a+\bar{\varepsilon}_\mu$, for a certain $\mu\,(0 \leqslant \mu \leqslant n-1)$. Non-zero Boltzmann weights are parametrized in terms of the elliptic theta function of the spectral parameter v as follows:

$$W\left[\begin{array}{cc|c} a+2\bar{\varepsilon}_\mu & a+\bar{\varepsilon}_\mu & \\ a+\bar{\varepsilon}_\mu & a & v \end{array}\right] = r_1(v), \tag{A.6}$$

$$W\left[\begin{array}{cc|c} a+\bar{\varepsilon}_\mu+\bar{\varepsilon}_\nu & a+\bar{\varepsilon}_\mu & \\ a+\bar{\varepsilon}_\nu & a & v \end{array}\right] = r_1(v)\frac{[v][a_{\mu\nu}+1]}{[1-v][a_{\mu\nu}]} \quad (\mu \neq \nu), \tag{A.7}$$

$$W\left[\begin{array}{cc|c} a+\bar{\varepsilon}_\mu+\bar{\varepsilon}_\nu & a+\bar{\varepsilon}_\mu & \\ a+\bar{\varepsilon}_\mu & a & v \end{array}\right] = r_1(v)\frac{[1][v+a_{\mu\nu}]}{[1-v][a_{\mu\nu}]} \quad (\mu \neq \nu), \tag{A.8}$$

where $r_1(v)$ is defined by (2.3). In this paper we consider so-called Regime III in the model, i.e., $0 < v < 1$.

B Appendix B OPE formulae and commutation relations

In this Appendix we list some useful formulae for the basic bosons. In what follows we denote $z = x^{2v}$, $z' = x^{2v'}$.

First, useful OPE formulae are:

$$U_{\omega_1}(v)U_{\omega_j}(v') = z^{\frac{r-1}{r}\frac{n-j}{n}} g_j(z'/z) : U_{\omega_1}(v)U_{\omega_j}(v') :, \tag{B.1}$$

$$U_{\omega_j}(v)U_{\omega_1}(v') = z^{\frac{r-1}{r}\frac{n-j}{n}} g_j(z'/z) : U_{\omega_j}(v)U_{\omega_1}(v') :, \tag{B.2}$$

$$U_{\omega_j}(v)U_{-\alpha_j}(v') = z^{-\frac{r-1}{r}} \frac{(x^{2r-1}z'/z;x^{2r})_\infty}{(xz'/z;x^{2r})_\infty} : U_{\omega_j}(v)U_{-\alpha_j}(v') :, \tag{B.3}$$

$$U_{-\alpha_j}(v)U_{\omega_j}(v') = z^{-\frac{r-1}{r}} \frac{(x^{2r-1}z'/z;x^{2r})_\infty}{(xz'/z;x^{2r})_\infty} : U_{-\alpha_j}(v)U_{\omega_j}(v') :, \tag{B.4}$$

$$U_{-\alpha_j}(v)U_{-\alpha_{j\pm1}}(v') = z^{-\frac{r-1}{r}} \frac{(x^{2r-1}z'/z;x^{2r})_\infty}{(xz'/z;x^{2r})_\infty} : U_{-\alpha_j}(v)U_{-\alpha_{j\pm1}}(v') :, \tag{B.5}$$

$$U_{-\alpha_j}(v)U_{-\alpha_j}(v') = z^{\frac{2(r-1)}{r}} \left(1-\frac{z'}{z}\right) \frac{(x^2z'/z;x^{2r})_\infty}{(x^{2r-2}z'/z;x^{2r})_\infty} : U_{-\alpha_j}(v)U_{-\alpha_j}(v') :, \tag{B.6}$$

$$V_{\omega_1}(v)V_{\omega_j}(v') = z^{\frac{r}{r-1}\frac{n-j}{n}} g_j^*(z'/z) : V_{\omega_1}(v)V_{\omega_j}(v') :, \tag{B.7}$$

$$V_{\omega_j}(v)V_{\omega_1}(v') = z^{\frac{r}{r-1}\frac{n-j}{n}} g_j^*(z'/z) : V_{\omega_j}(v)V_{\omega_1}(v') :, \tag{B.8}$$

$$V_{\omega_j}(v)V_{-\alpha_j}(v') = z^{-\frac{r}{r-1}} \frac{(x^{2r-1}z'/z;x^{2r-2})_\infty}{(x^{-1}z'/z;x^{2r-2})_\infty} : V_{\omega_j}(v)V_{-\alpha_j}(v') :, \tag{B.9}$$

$$V_{-\alpha_j}(v)V_{\omega_j}(v') = z^{-\frac{r}{r-1}} \frac{(x^{2r-1}z'/z;x^{2r-2})_\infty}{(x^{-1}z'/z;x^{2r-2})_\infty} : V_{-\alpha_j}(v)V_{\omega_j}(v') :, \tag{B.10}$$

$$V_{-\alpha_j}(v)V_{-\alpha_{j\pm1}}(v') = z^{-\frac{r}{r-1}} \frac{(x^{2r-1}z'/z;x^{2r-2})_\infty}{(x^{-1}z'/z;x^{2r-2})_\infty} : V_{-\alpha_j}(v)V_{-\alpha_{j\pm1}}(v') :, \tag{B.11}$$

$$V_{-\alpha_j}(v)V_{-\alpha_j}(v') = z^{\frac{2r}{r-1}} \left(1-\frac{z'}{z}\right) \frac{(x^{-2}z'/z;x^{2r-2})_\infty}{(x^{2r}z'/z;x^{2r-2})_\infty} : V_{-\alpha_j}(v)V_{-\alpha_j}(v') :, \tag{B.12}$$

$$V_{\omega_j}(v)U_{\omega_j}(v') = z^{-\frac{j(n-j)}{n}} \rho_j(z'/z) : V_{\omega_1}(v)U_{\omega_j}(v') :, \tag{B.13}$$

$$U_{\omega_j}(v)V_{\omega_j}(v') = z^{-\frac{j(n-j)}{n}} \rho_j(z'/z) : U_{\omega_j}(v)V_{\omega_j}(v') :, \tag{B.14}$$

$$V_{\omega_j}(v)U_{-\alpha_j}(v') = z\left(1+\frac{z'}{z}\right) : V_{\omega_j}(v)U_{-\alpha_j}(v') := U_{-\alpha_j}(v')V_{\omega_j}(v), \tag{B.15}$$

$$U_{\omega_j}(v)V_{-\alpha_j}(v') = z\left(1+\frac{z'}{z}\right) : U_{\omega_j}(v)V_{-\alpha_j}(v') := V_{-\alpha_j}(v')U_{\omega_j}(v), \tag{B.16}$$

$$V_{-\alpha_j}(v)U_{-\alpha_{j\pm1}}(v') = z\left(1+\frac{z'}{z}\right) : V_{-\alpha_j}(v)U_{-\alpha_{j\pm1}}(v') := U_{-\alpha_{j\pm1}}(v')V_{-\alpha_j}(v), \tag{B.17}$$

$$V_{-\alpha_j}(v)U_{-\alpha_j}(v') = \frac{: V_{-\alpha_j}(v)U_{-\alpha_j}(v') :}{z^2(1+\frac{xz'}{z})(1+\frac{x^{-1}z'}{z})}, \tag{B.18}$$

$$U_{-\alpha_j}(v)V_{-\alpha_j}(v') = \frac{: U_{-\alpha_j}(v)V_{-\alpha_j}(v') :}{z^2(1+\frac{xz'}{z})(1+\frac{x^{-1}z'}{z})}, \tag{B.19}$$

where $g_j(z)$, $g_j^*(z)$ and $\rho_j(z)$ are defined by (2.3), (2.4) and (2.5). From these, we obtain the following commutation relations:

$$U_{\omega_1}(v)U_{\omega_j}(v') = r_j(v-v')U_{\omega_j}(v')U_{\omega_1}(v), \tag{B.20}$$

$$U_{-\alpha_j}(v)U_{\omega_j}(v') = -f(v-v',0)U_{\omega_j}(v')U_{-\alpha_j}(v), \tag{B.21}$$

$$U_{-\alpha_j}(v)U_{-\alpha_{j\pm1}}(v') = -f(v-v',0)U_{-\alpha_{j\pm1}}(v')U_{-\alpha_j}(v), \tag{B.22}$$

$$U_{-\alpha_j}(v)U_{-\alpha_j}(v') = h(v-v')U_{-\alpha_j}(v')U_{-\alpha_j}(v), \tag{B.23}$$

$$V_{\omega_1}(v)V_{\omega_j}(v') = r_j^*(v-v')V_{\omega_j}(v')V_{\omega_1}(v), \tag{B.24}$$

$$V_{-\alpha_j}(v)V_{\omega_j}(v') = -f^*(v-v',0)V_{\omega_j}(v')V_{-\alpha_j}(v), \tag{B.25}$$

$$V_{-\alpha_j}(v)V_{-\alpha_{j\pm1}}(v') = -f^*(v-v',0)V_{-\alpha_{j\pm1}}(v')V_{-\alpha_j}(v), \tag{B.26}$$

$$V_{-\alpha_j}(v)V_{-\alpha_j}(v') = h^*(v-v')V_{-\alpha_j}(v')V_{-\alpha_j}(v), \tag{B.27}$$

$$U_{\omega_j}(v)V_{\omega_j}(v') = \chi_j(v-v')V_{\omega_j}(v')U_{\omega_j}(v), \tag{B.28}$$

$$[V_{-\alpha_j}(v), U_{-\alpha_j}(v')] = \frac{\delta(\frac{z}{-xz'})-\delta(\frac{z'}{-xz})}{(x^{-1}-x)zz'} : V_{-\alpha_j}(v)U_{-\alpha_j}(v') :, \tag{B.29}$$

where $r_j(v)$, $r_j^*(v)$, $\chi_j(v)$, $f(v,w)$, $h(v)$, $f^*(v,w)$ and $h^*(v)$ are defined by (2.3), (2.4), (2.5), (2.7) and (2.8), and the δ-function is defined by the following formal power series

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n.$$

The commutation relation (B.29) can be derived from (B.18), (B.19) and the identity

$$\frac{1}{z^2(1+\frac{xz'}{z})(1+\frac{x^{-1}z'}{z})} - \frac{1}{z'^2(1+\frac{xz}{z'})(1+\frac{x^{-1}z}{z'})} = \frac{\delta(\frac{z}{-xz'}) - \delta(\frac{z'}{-xz})}{(x^{-1}-x)zz'}.$$

The relation (B.29) can be practically understood as follows. Let us compare the integrals

$$\oint \frac{dz}{2\pi\sqrt{-1}} V_{-\alpha_j}(v) U_{-\alpha_j}(v') F(v,v'), \tag{B.30}$$

and

$$\oint \frac{dz}{2\pi\sqrt{-1}} U_{-\alpha_j}(v') V_{-\alpha_j}(v) F(v,v'), \tag{B.31}$$

where $F(u,v)$ is an appropriate function. Note that the normal order product expansion (B.18) is valid for $|z| > |-x^{\pm1}z'|$ while (B.19) is valid for $|z'| > |-x^{\pm1}z|$. Thus, the integral contour of (B.30) encircles the poles $-x^{\pm1}z'$, but that of (B.31) does not encircle them. The difference between (B.30) and (B.31) can be therefore evaluated by the residues at $z = -x^{\pm1}z'$.

Finally, we list the OPE formulae for $W_{-\alpha_j}(v)$ and other basic operators:

$$\begin{aligned}
W_{-\alpha_j}(v)V_{-\alpha_{j\pm1}}(v') &= -(-z)^{-\frac{1}{r-1}} \frac{(-x^r z'/z; x^{2r-2})_\infty}{(-x^{r-2}z'/z; x^{2r-2})_\infty} : W_{-\alpha_j}(v)V_{-\alpha_{j\pm1}}(v') :, && \text{(B.32)}\\
V_{-\alpha_{j\pm1}}(v)W_{-\alpha_j}(v') &= z^{-\frac{1}{r-1}} \frac{(-x^r z'/z; x^{2r-2})_\infty}{(-x^{r-2}z'/z; x^{2r-2})_\infty} : V_{-\alpha_{j\pm1}}(v)W_{-\alpha_j}(v') :, && \text{(B.33)}\\
V_{\omega_j}(v)W_{-\alpha_j}(v') &= z^{-\frac{1}{r-1}} \frac{(-x^r z'/z; x^{2r-2})_\infty}{(-x^{r-2}z'/z; x^{2r-2})_\infty} : V_{\omega_j}(v)W_{-\alpha_j}(v') :, && \text{(B.34)}\\
W_{-\alpha_j}(v)V_{\omega_j}(v') &= -(-z)^{-\frac{1}{r-1}} \frac{(-x^r z'/z; x^{2r-2})_\infty}{(-x^{r-2}z'/z; x^{2r-2})_\infty} : W_{-\alpha_j}(v)V_{\omega_j}(v') :, && \text{(B.35)}
\end{aligned}$$

$$\begin{aligned}
U_{-\alpha_{j\pm1}}(v)W_{-\alpha_j}(v') &= z^{\frac{1}{r}} \frac{(x^{r-1}z'/z; x^{2r})_\infty}{(x^{r+1}z'/z; x^{2r})_\infty} : U_{-\alpha_{j\pm1}}(v)W_{-\alpha_j}(v') :, && \text{(B.36)}\\
W_{-\alpha_j}(v)U_{-\alpha_{j\pm1}}(v') &= -z^{\frac{1}{r}} \frac{(x^{r-1}z'/z; x^{2r})_\infty}{(x^{r+1}z'/z; x^{2r})_\infty} : W_{-\alpha_j}(v)U_{-\alpha_{j\pm1}}(v') :, && \text{(B.37)}\\
U_{\omega_j}(v)W_{-\alpha_j}(v') &= z^{\frac{1}{r}} \frac{(x^{r-1}z'/z; x^{2r})_\infty}{(x^{r+1}z'/z; x^{2r})_\infty} : U_{\omega_j}(v)W_{-\alpha_j}(v') :, && \text{(B.38)}\\
W_{-\alpha_j}(v)U_{\omega_j}(v') &= -z^{\frac{1}{r}} \frac{(x^{r-1}z'/z; x^{2r})_\infty}{(x^{r+1}z'/z; x^{2r})_\infty} : W_{-\alpha_j}(v)U_{\omega_j}(v') : . && \text{(B.39)}
\end{aligned}$$

From these, we obtain

$$\begin{aligned}
W_{-\alpha_j}\left(v + \tfrac{r}{2} - \tfrac{\pi\sqrt{-1}}{2\epsilon}\right) V_{-\alpha_{j\pm1}}(v) &= 0 = V_{-\alpha_{j\pm1}}(v) W_{-\alpha_j}\left(v - \tfrac{r}{2} - \tfrac{\pi\sqrt{-1}}{2\epsilon}\right), && \text{(B.40)}\\
W_{-\alpha_j}\left(v + \tfrac{r}{2} - \tfrac{\pi\sqrt{-1}}{2\epsilon}\right) V_{\omega_j}(v) &= 0 = V_{\omega_j}(v) W_{-\alpha_j}\left(v - \tfrac{r}{2} - \tfrac{\pi\sqrt{-1}}{2\epsilon}\right), && \text{(B.41)}\\
U_{-\alpha_{j\pm1}}(v) W_{-\alpha_j}\left(v - \tfrac{r-1}{2}\right) &= 0 = W_{-\alpha_j}\left(v + \tfrac{r-1}{2}\right) U_{-\alpha_{j\pm1}}(v), && \text{(B.42)}\\
U_{\omega_j}(v) W_{-\alpha_j}\left(v - \tfrac{r-1}{2}\right) &= 0 = W_{-\alpha_j}\left(v + \tfrac{r-1}{2}\right) U_{\omega_j}(v). && \text{(B.43)}
\end{aligned}$$

C Appendix C Commutation relations of $\Phi(u)_a^{a'}$ and $\Psi^*(v)_\xi^{\xi'}$

In this appendix, we give a remark on the commutation relation (3.14). In [14] we proved (3.14) on the assumption of the commutativity of $U_{-\alpha_j}(v)$ and $V_{-\alpha_j}(v')$. From (B.29), however, $U_{-\alpha_j}(v)$ and $V_{-\alpha_j}(v')$ commute at all points but at $v' = v \pm \frac{1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}$. Nevertheless, (3.14) holds, which we will briefly show in this appendix.

Let $a' - a = \bar{\varepsilon}_\mu$ and $\xi' - \xi = \bar{\varepsilon}_\nu$ on (3.14). We assume that $\mu \leqslant \nu$. (The case $\mu > \nu$ can be similarly proved.) When $\mu = 0$, (3.14) follows from (B.15–B.17) and (B.28). When $\mu = 1$, the difference of the both sides of (3.14) can be calculated as follows:

$$\begin{aligned} & \Phi(u)_a^{a+\bar{\varepsilon}_1}\Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_\nu} - \chi(u-v)\Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_\nu}\Phi(v)_a^{a+\bar{\varepsilon}_1} \\ = \; & U_{\omega_1}(u)V_{\omega_1}(v)\oint_C \frac{dz_1'}{2\pi\sqrt{-1}z_1'}\oint_{C'}\prod_{j=1}^{\nu}\frac{dz_j}{2\pi\sqrt{-1}z_j}[U_{-\alpha_1}(u_1), V_{-\alpha_1}(v_1)] \\ \times \; & V_{-\alpha_2}(v_2)\cdots V_{-\alpha_\nu}(v_\nu) f(u_1-u, K_{01})\prod_{\substack{j=0\\ j\neq 1}}^{n-1}[K_{j1}]^{-1}\prod_{j=0}^{\nu-1} f^*(v_{j+1}-v_j, L_{j\nu}), \end{aligned} \tag{C.1}$$

where $z_j = x^{2v_j}$ and $z_j' = x^{2u_j}$. From (B.29) the integral with respect to z_1 of (C.1) can be evaluated by the residues at $z_1 = -x^{\pm 1}z_1'$. Repeating similar calculations performed in section 4.4, the RHS of (C.1) can be rewritten as a total difference of such a form

$$U_{\omega_1}(u)V_{\omega_1}(v)\left(\oint_{x^{-r}C} - \oint_{x^r C}\right)\frac{dz_1'}{2\pi\sqrt{-1}z_1'}\oint_{C'}\prod_{j=2}^{\nu}\frac{dz_j}{2\pi\sqrt{-1}z_j}W_{-\alpha_1}(u_1)V_{-\alpha_2}(v_2)\cdots V_{-\alpha_\nu}(v_\nu)G(u_1), \tag{C.2}$$

where

$$G\left(u_1 + \tfrac{r}{2}\right) = \frac{1}{x^{-1}-x} f(u_1-u, a_{01})\prod_{\substack{j=0\\ j\neq 1}}^{n-1}[a_{j1}]^{-1}\prod_{j=0}^{\nu-1} f^*(v_{j+1}-v_j, \xi_{j\nu})\Bigg|_{v_1 = u_1 + \frac{1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}}.$$

In the present case, there are at most three poles at $u_1 = u - \frac{r-1}{2}, v - \frac{r}{2} - \frac{\pi\sqrt{-1}}{2\epsilon}, v_2 + \frac{r}{2} - \frac{\pi\sqrt{-1}}{2\epsilon}$, inside the contour for z_1'-integration. The residues at those three points vanish because of (B.43), (B.41), and (B.40), respectively. Therefore we have

$$\Phi(u)_a^{a+\bar{\varepsilon}_1}\Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_\nu} - \chi(u-v)\Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_\nu}\Phi(v)_a^{a+\bar{\varepsilon}_1} = 0.$$

When $\mu \geqslant 2$, the difference of the both sides of (3.14) can be calculated as follows:

$$\Phi(u)_a^{a+\bar{\varepsilon}_\mu}\Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_\nu} - \chi(u-v)\Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_\nu}\Phi(v)_a^{a+\bar{\varepsilon}_\mu} = \sum_{\lambda=1}^{\mu}\oint_C\prod_{j=1}^{\mu}\frac{dz_j'}{2\pi\sqrt{-1}z_j'}\oint_{C'}\prod_{j'=1}^{\nu}\frac{dz_{j'}}{2\pi\sqrt{-1}z_{j'}}X_\lambda, \tag{C.3}$$

where

$$\begin{aligned} X_\lambda = \; & U_{\omega_1}(u)V_{\omega_1}(v)V_{-\alpha_1}(v_1)U_{-\alpha_1}(v_1)\cdots V_{-\alpha_{\lambda-1}}(v_{\lambda-1})U_{-\alpha_{\lambda-1}}(u_{\lambda-1})[U_{-\alpha_\lambda}(u_\lambda), V_{-\alpha_\lambda}(v_\lambda)] \\ \times \; & U_{-\alpha_{\lambda+1}}(u_{\lambda+1})V_{-\alpha_{\lambda+1}}(v_{\lambda+1})\cdots U_{-\alpha_\mu}(u_\mu)V_{-\alpha_\mu}(v_\mu)\cdots V_{-\alpha_\nu}(v_\nu) \\ \times \; & \prod_{j=0}^{\mu-1} f(u_{j+1}-u_j, K_{j\mu})\prod_{\substack{j=0\\ j\neq\mu}}^{n-1}[K_{j\mu}]^{-1}\prod_{j'=0}^{\nu-1} f^*(v_{j'+1}-v_{j'}, L_{j'\nu}). \end{aligned} \tag{C.4}$$

From (B.29) the integral with respect to z_λ of X_λ can be evaluated by the residues at $z_\lambda = -x^{\pm 1}z'_\lambda$. Similarly to (C.2), the result can be rewritten as a total difference of such a form

$$\begin{aligned}
&\oint_C \prod_{j=1}^{\mu} \frac{dz'_j}{2\pi\sqrt{-1}z'_j} \oint_{C'} \prod_{j'=1}^{\nu} \frac{dz_{j'}}{2\pi\sqrt{-1}z_{j'}} X_\lambda = U_{\omega_1}(u)V_{\omega_1}(v)\left(\oint_{x^{-r}C} - \oint_{x^r C}\right)\frac{dz'_\lambda}{2\pi\sqrt{-1}z'_\lambda} \\
\times\;& \oint_C \prod_{\substack{j=1\\ j\neq\lambda}}^{\mu} \frac{dz'_j}{2\pi\sqrt{-1}z'_j} \oint_{C'} \prod_{\substack{j'=1\\ j'\neq\lambda}}^{\nu} \frac{dz_{j'}}{2\pi\sqrt{-1}z_{j'}} V_{-\alpha_1}(v_1)U_{-\alpha_1}(v_1)\cdots V_{-\alpha_{\lambda-1}}(v_{\lambda-1})U_{-\alpha_{\lambda-1}}(u_{\lambda-1}) \qquad (C.5)\\
\times\;& W_{-\alpha_\lambda}(u_\lambda)U_{-\alpha_{\lambda+1}}(u_{\lambda+1})V_{-\alpha_{\lambda+1}}(v_{\lambda+1})\cdots U_{-\alpha_\mu}(u_\mu)V_{-\alpha_\mu}(v_\mu)\cdots V_{-\alpha_\nu}(v_\nu)G_\lambda(u_\lambda),
\end{aligned}$$

where

$$G_\lambda\left(u_\lambda + \tfrac{r}{2}\right) = \frac{1}{x^{-1}-x}\prod_{j=0}^{\mu-1} f(u_{j+1}-u_j, a_{j\mu}) \prod_{\substack{j=0\\ j\neq\mu}}^{n-1} [a_{j\mu}]^{-1} \prod_{j'=0}^{\nu-1} f^*(v_{j'+1}-v_{j'}, \xi_{j'\nu})\Bigg|_{v_\lambda = u_\lambda + \frac{1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}}.$$

In the present case, there are at most four poles at $u_\lambda = u_{\lambda\pm1} \pm \frac{r-1}{2}, v_{\lambda\pm1} \pm \frac{r}{2} - \frac{\pi\sqrt{-1}}{2\epsilon}$, inside the contour for z'_λ-integration. The residues at those four points vanish because of (B.42) and (B.40), respectively. (When $\lambda = 1$, we also use (B.43) and (B.41) as well as (B.42) and (B.40).) Therefore we prove (3.14) for $\mu \geqslant 2$.

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